

Recursion Operators and Nonlocal Symmetries for Integrable rmdKP and rdDym Equations

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Abstract. We find direct and inverse recursion operators for integrable cases of the rmdKP and rdDym equations. Also, we study actions of these operators on the contact symmetries and find shadows of nonlocal symmetries of these equations.

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1. Introduction

The interrelated notions of infinite hierarchies of symmetries and conservation laws, recursion operators, bi-Hamiltonian structures, and differential coverings are the main tools in the study of integrable nonlinear partial differential equations (PDEs), [32, 36, 20, 13, 18, 21, 14, 15, 3, 33, 1]. In particular, a recursion operator for a PDE is a linear map from the space of symmetries of the PDE to the same space. Procedures to find recursion operators have been proposed by many authors, see, e.g., [32, 9, 51, 7, 6, 40, 8, 45, 3, 38, 39, 33, 11, 10, 16, 17, 24, 12, 46, 44, 50, 29, 25, 26, 47, 48, 23, 28, 27, 41, 19, 22, 30]. As a rule, recursion operators are nonlocal. This is one of the reasons motivating the introduction of nonlocal symmetries and, more generally, the development of nonlocal geometry of PDEs, [20, 18, 21]. In a majority of works recursion operators are defined as integro-differential operators, [32, 9, 51, 7, 6, 8, 45, 3, 33, 12, 46, 44, 50], although this interpretation is accompanied by a number of difficulties, e.g., discussed in [10, 20]. The alternative definition is proposed in [10], [16, 17] (see also [41] and references therein) and developed in [24, 46, 29, 25, 26, 28, 27, 30]. This approach considers a recursion operator as an auto-Bäcklund transformation of the tangent (or linearized) covering of the PDE. The machinery of recursion operators become more difficult when transitioning from PDEs with two independent variables to multidimensional PDEs. Accordingly, only a small number of recursion operators for PDEs in three or more independent variables is currently known. In [30], M. Marvan and A. Sergyeyev proposed the method for constructing recursion operators of PDEs of any dimension from their linear coverings of a special form. By this method they found recursion operators for a number of PDEs of physical and geometrical significance.

In the present paper we consider PDEs

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy} \quad (1)$$

and

$$u_{ty} = u_x u_{xy} - u_y u_{xx}. \quad (2)$$

Eq. (1) describes Lorentzian hyperCR Einstein-Weyl structures and is a symmetry reduction of Plebański's second heavenly equation, [5]. It belongs to the family of r -th modified dispersionless Kadomtsev–Petviashvili equations (rmdKP) [2],

$$u_{yy} = u_{tx} + \left(\frac{1}{2}(\kappa + 1)u_x^2 + u_y\right)u_{xx} + \kappa u_x u_{xy}.$$

Eq. (2) is obtained by substituting for $\kappa = -1$ to the family of r -th dispersionless $(2+1)$ -dimensional Harry Dym equation (rdDym) [2],

$$u_{ty} = u_x u_{xy} + \kappa u_y u_{xx}.$$

Both Eqs. (1) and (2) are known to have two attributes of an integrable PDE. They are bi-Hamiltonian systems on two-dimensional generalizations of the Virasoro algebra,

[34, 35]. Also, they have differential coverings with non-removable parameters. The covering

$$\begin{cases} w_t &= (\lambda^2 - \lambda u_x - u_y) w_x, \\ w_y &= (\lambda - u_x) w_x, \end{cases} \quad (3)$$

$\lambda \in \mathbb{R}$, for Eq. (1) was found in [42, 5], the covering

$$\begin{cases} w_t &= (u_x - \lambda) w_x, \\ w_y &= \lambda^{-1} u_y w_x, \end{cases} \quad (4)$$

of Eq. (2) was found for $\lambda = 1$ in [43] and for $\lambda \in \mathbb{R} \setminus \{0\}$ in [31].

Also, a hereditary recursion operator for Eq. (1) was found in [23].

In the present paper, we use the technique of [30] to construct recursion operators for Eqs. (1) and (2). Section 2 is devoted to notation and basic definitions of the geometry PDES, [49, 20, 18, 21, 19]. Section 3 recalls the method of [30]. In Section 4 we find the direct and inverse recursion operators for (1) and (2). In Section 5 we study actions of these operators on the contact symmetries of (1), (2) and find shadows of nonlocal symmetries of these equations.

2. Preliminaries

Let $\pi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\pi: (x^1, \dots, x^n, u^1, \dots, u^m) \mapsto (x^1, \dots, x^n)$, be a trivial bundle, and $J^\infty(\pi)$ be the bundle of its jets of the infinite order. The local coordinates on $J^\infty(\pi)$ are $(x^i, u^\alpha, u_I^\alpha)$, where $I = (i_1, \dots, i_n)$ is a multi-index, and for every local section $f: \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ of π the corresponding infinite jet $j_\infty(f)$ is a section $j_\infty(f): \mathbb{R}^n \rightarrow J^\infty(\pi)$ such that $u_I^\alpha(j_\infty(f)) = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\dots+i_n} f^\alpha}{(\partial x^1)^{i_1} \dots (\partial x^n)^{i_n}}$. We put $u^\alpha = u_{(0,\dots,0)}^\alpha$. Also, in the case of $n = 3$, $m = 1$ we denote $x^1 = t$, $x^2 = x$, $x^3 = y$, and $u_{(i,j,k)}^1 = \underbrace{u_{t \dots t}}_i \underbrace{x \dots x}_j \underbrace{y \dots y}_k$.

The the vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m u_{I+1_k}^\alpha \frac{\partial}{\partial u_I^\alpha}, \quad k \in \{1, \dots, n\},$$

$(i_1, \dots, i_k, \dots, i_n) + 1_k = (i_1, \dots, i_k + 1, \dots, i_n)$, are called *total derivatives*. They commute everywhere on $J^\infty(\pi)$: $[D_{x^i}, D_{x^j}] = 0$.

The *evolutionary differentiation* associated to an arbitrary vector-valued smooth function $\varphi: J^\infty(\pi) \rightarrow \mathbb{R}^m$ is the vector field

$$\mathbf{E}_\varphi = \sum_{\#I \geq 0} \sum_{\alpha=1}^m D_I(\varphi^\alpha) \frac{\partial}{\partial u_I^\alpha}, \quad (5)$$

with $D_I = D_{(i_1, \dots, i_n)} = D_{x^1}^{i_1} \circ \dots \circ D_{x^n}^{i_n}$.

A system of PDEs $F_r(x^i, u_I^\alpha) = 0$, $\#I \leq s$, $r \in \{1, \dots, R\}$ of the order $s \geq 1$ with $R \geq 1$, defines the submanifold $\mathcal{E} = \{(x^i, u_I^\alpha) \in J^\infty(\pi) \mid D_K(F_r(x^i, u_I^\alpha)) = 0, \#K \geq 0\}$ in $J^\infty(\pi)$.

A function $\varphi: J^\infty(\pi) \rightarrow \mathbb{R}^m$ is called a (*generator of an infinitesimal*) *symmetry* of \mathcal{E} when $\mathbf{E}_\varphi(F) = 0$ on \mathcal{E} . The symmetry φ is a solution to the *defining system*

$$\ell_{\mathcal{E}}(\varphi) = 0, \quad (6)$$

where $\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}$ with the matrix differential operator

$$\ell_F = \left(\sum_{\#I \geq 0} \frac{\partial F_r}{\partial u_I^\alpha} D_I \right)$$

The *symmetry algebra* $\text{sym}(\mathcal{E})$ consists of solutions to (6). The *Jacobi bracket* $\{\varphi, \psi\} = \mathbf{E}_\varphi(\psi) - \mathbf{E}_\psi(\varphi)$ defines a structure of a Lie algebra over \mathbb{R} on $\text{sym}\mathcal{E}$. The *algebra of contact symmetries* $\text{sym}_0(\mathcal{E})$ is the Lie subalgebra of $\text{sym}(\mathcal{E})$ defined as $\text{sym}(\mathcal{E}) \cap J^1(\pi)$.

A *conservation law* of \mathcal{E} is an equivalence class of $(n-1)$ -forms

$$\omega = \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} b_{i_1 \dots i_{n-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}}$$

with $b_{i_1 \dots i_{n-1}} \in C^\infty(J^\infty(\pi))$ such that

$$d_h \omega = \sum_{k=1}^n \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} D_k(b_{i_1 \dots i_{n-1}}) dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}} = 0$$

on \mathcal{E} . Two such forms are equivalent when their difference is a form

$$\theta = \sum_{k=1}^n \sum_{1 \leq i_1 < \dots < i_{n-2} \leq n} D_k(c_{i_1 \dots i_{n-2}}) dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{n-2}},$$

so $d_h \theta = 0$.

Denote $\mathcal{W} = \mathbb{R}^\infty$ with coordinates w^s , $s \in \mathbb{N} \cup \{0\}$. Locally, an (infinite-dimensional) *differential covering* of \mathcal{E} is a trivial bundle $\tau: J^\infty(\pi) \times \mathcal{W} \rightarrow J^\infty(\pi)$ equipped with *extended total derivatives*

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^{\infty} T_k^s(x^i, u_I^\alpha, w^j) \frac{\partial}{\partial w^s} \quad (7)$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \neq j$ whenever $(x^i, u_I^\alpha) \in \mathcal{E}$. We define the partial derivatives of w^s by $w_{x^k}^s = \tilde{D}_{x^k}(w^s)$. This yields the system of *covering equations*

$$w_{x^k}^s = T_k^s(x^i, u_I^\alpha, w^j). \quad (8)$$

This over-determined system of PDEs is compatible whenever $(x^i, u_I^\alpha) \in \mathcal{E}$.

EXAMPLE 1. In the case of $n = 3$ and $m = 1$ consider the extended total derivatives

$$\begin{cases} \tilde{D}_t &= D_t + \sum_{k=0}^{\infty} \tilde{D}_x^k((u_x - \lambda) w_1) \frac{\partial}{\partial w_k}, \\ \tilde{D}_x &= D_x + \sum_{k=0}^{\infty} w_{k+1} \frac{\partial}{\partial w_k}, \\ \tilde{D}_y &= D_y + \lambda^{-1} \sum_{k=0}^{\infty} \tilde{D}_x^k(u_y w_1) \frac{\partial}{\partial w_k}. \end{cases} \quad (9)$$

Then define the partial derivatives of the fiber variables as $w_{k,t} = \tilde{D}_t(w_k)$, $w_{k,x} = \tilde{D}_x(w_k)$, and $w_{k,y} = \tilde{D}_y(w_k)$. This implies $w_k = w_{0,x\dots x}$ (k times x), and

$$\begin{cases} w_{k,t} &= ((u_x - \lambda) w_1)_{x\dots x}, \\ w_{k,y} &= \lambda^{-1} (u_y w_1)_{x\dots x}. \end{cases}$$

All these equations are differential consequences of the system

$$\begin{cases} w_{0,t} &= (u_x - \lambda) w_{0,x}, \\ w_{0,y} &= \lambda^{-1} u_y w_{0,x}. \end{cases}$$

We put $w_0 = w$ and get Eqs. (4). Thus the covering with the extended total derivatives (9) is defined by (4).

Denote by $\tilde{\mathbf{E}}_\varphi$ the result of substitution for \tilde{D}_{x^k} instead of D_{x^k} in (5). A *shadow of nonlocal symmetry* of \mathcal{E} corresponding to the covering τ with the extended total derivatives (7), or τ -*shadow*, is a function $\varphi \in C^\infty(\mathcal{E} \times \mathcal{W})$ such that

$$\tilde{\mathbf{E}}_\varphi(F) = 0 \quad (10)$$

is a consequence of equations $D_K(F) = 0$ and (8). A *nonlocal symmetry* of \mathcal{E} corresponding to the covering τ (or τ -*symmetry*) is the vector field

$$\tilde{\mathbf{E}}_{\varphi,A} = \tilde{\mathbf{E}}_\varphi + \sum_{s=0}^{\infty} A^s \frac{\partial}{\partial w_s}, \quad (11)$$

with $A^s \in C^\infty(\mathcal{E} \times \mathcal{W})$ such that φ satisfies to (10) and

$$\tilde{D}_k(A^s) = \tilde{\mathbf{E}}_{\varphi,A}(T_k^s) \quad (12)$$

for T_k^s from (7), see [4, Ch. 6, §3.2].

REMARK 1. In general, not every τ -shadow corresponds to a τ -symmetry, since Eqs. (12) provide an obstruction for existence of (11). But for any τ -shadow φ there exists a covering τ_φ and a nonlocal τ_φ -symmetry whose τ_φ -shadow coincides with φ , see [4, Ch. 6, §5.8].

A *recursion operator* \mathcal{R} for \mathcal{E} is a \mathbb{R} -linear map such that for each (local or nonlocal) symmetry φ of \mathcal{E} the function $\mathcal{R}(\varphi)$ is a (local or nonlocal) symmetry of φ of \mathcal{E} .

The tangent covering for PDE \mathcal{E} is defined as follows, [19]. Consider the trivial bundle $\sigma: J^\infty(\pi) \times \mathcal{Q} \rightarrow J^\infty(\pi)$ with coordinates q_I^α , $\#I \geq 0$, on the fibre \mathcal{Q} equipped with the extended total derivatives

$$\hat{D}_{x^k} = D_{x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m q_{I+1^k}^\alpha \frac{\partial}{\partial q_I^\alpha}.$$

Then for $\hat{D}_I = \hat{D}_{x^1}^{i_1} \circ \dots \circ \hat{D}_{x^n}^{i_n}$ define

$$\hat{\ell}_F = \left(\sum_{\#I \geq 0} \frac{\partial F_r}{\partial u_I^\alpha} \hat{D}_I \right).$$

and put

$$\mathcal{T}(\mathcal{E}) = \{(x^i, u_i^\alpha, q_I^\alpha) \in J^\infty(\pi) \times \mathcal{Q} \mid D_K(F(x^i, u_i^\alpha)) = 0, \hat{D}_K(\hat{\ell}_F(q^\alpha)) = 0, \#K \geq 0\}.$$

The *tangent covering* is the restriction of σ to $\mathcal{T}(\mathcal{E})$. A section $\varphi: \mathcal{E} \rightarrow \mathcal{T}(\mathcal{E})$ of the tangent covering is a symmetry of \mathcal{E} . The extended total derivatives of this covering are $\tilde{D}_{x^k} = \hat{D}_{x^k}|_{\mathcal{T}(\mathcal{E})}$.

EXAMPLE 2. We write Eq. (2) in the form $u_{ty} - u_x u_{xy} + u_y u_{xx} = 0$. Then we have

$$\ell_F(\varphi) = D_t D_y(\varphi) - u_x D_x D_y(\varphi) - u_{xy} D_x(\varphi) + u_y D_x^2(\varphi) + u_{xx} D_y(\varphi)$$

and

$$\hat{\ell}_F(q) = q_{(1,0,1)} - u_x q_{(0,1,1)} - u_{xy} q_{(0,1,0)} + u_y q_{(0,2,0)} + u_{xx} q_{(0,0,1)}.$$

The fiber of the tangent covering has local coordinates $q_{(i,j,0)}$ and $q_{(0,j,k)}$. The extended total derivatives of the tangent covering are

$$\left\{ \begin{array}{l} \tilde{D}_t = D_t + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{(i+1,j,0)} \frac{\partial}{\partial q_{(i,j,0)}} + \sum_{j=0}^{\infty} q_{(1,j,0)} \frac{\partial}{\partial q_{(0,j,0)}} \\ \quad + \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \tilde{D}_y^{k-1} (u_x q_{(0,1,1)} + u_{xy} q_{(0,1,0)} - u_y q_{(0,2,0)} - u_{xx} q_{(0,0,1)}) \frac{\partial}{\partial q_{(0,j,k)}}, \\ \tilde{D}_x = D_x + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{(i,j+1,0)} \frac{\partial}{\partial q_{(i,j,0)}} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_{(0,j+1,k)} \frac{\partial}{\partial q_{(0,j,k)}}, \\ \tilde{D}_y = D_y + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{D}_t^{i-1} (u_x q_{(0,1,1)} + u_{xy} q_{(0,1,0)} - u_y q_{(0,2,0)} - u_{xx} q_{(0,0,1)}) \frac{\partial}{\partial q_{(i,j,0)}} \\ \quad + \sum_{j=0}^{\infty} q_{(0,j,1)} \frac{\partial}{\partial q_{(0,j,0)}} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_{(0,j,k+1)} \frac{\partial}{\partial q_{(0,j,k)}}. \end{array} \right.$$

REMARK 2. Abusing the notation, we write $D_t^i D_x^j D_y^k(q)$ instead of $q_{(i,j,k)}$ in what follows.

3. Recursion operators for equations with linear coverings

The technique of [30] is applicable to PDEs with linear coverings defined by covering equations of the form

$$\sum_{i=1}^n A_j^i w_{x^i} = 0, \quad j \in \{1, 2\}, \quad (13)$$

with $A_j^i \in C^\infty(J^\infty(\pi))$. The commutativity condition for the corresponding vector fields

$$X_j = \sum_{i=1}^n A_j^i D_{x^i}$$

coincides with $D_K(F) = 0$, $\#K \geq 0$. One of the key elements of the method of [30] is the vector field

$$Z = \sum_{i=1}^n \zeta^i D_{x^i}$$

with $\zeta^i \in C^\infty(J^\infty(\pi))$ such that

$$[X_j, Z] = \sum_{i=1}^n \mathbf{E}_\varphi(A_j^i) D_{x^i}, \quad j \in \{1, 2\}, \quad (14)$$

with $\varphi \in C^\infty(J^\infty(\pi))$. The pair of equations (14) give an over-determined system for the functions ζ^i . This system is compatible whenever φ is a symmetry of \mathcal{E} . Then we seek for a set of functions a_1, \dots, a_n either from $C^\infty(J^\infty(\pi))$ or from $C^\infty(J^\infty(\pi) \times \mathcal{W})$ such that the function

$$\psi = \sum_{j=1}^n a_j \zeta^j \quad (15)$$

is either a local symmetry of \mathcal{E} or a shadow of a nonlocal symmetry of \mathcal{E} corresponding to the covering (13). Since ζ^j depend on φ , then (15) defines a map $\psi = \mathcal{R}(\varphi)$. This map is a recursion operator for \mathcal{E} .

4. Recursion operators for the the integrable rmdKP and rdDym equations

4.1. rmdKP equation

We apply the described above method of [30] to the covering (3) of Eq. (1). The straightforward computation shows that condition (14) holds if, and only if, the functions ζ_1 and ζ_3 are solutions to the following over-determined system

$$\begin{cases} D_t(\zeta) &= (\lambda^2 - \lambda u_x - u_y) D_x(\zeta), \\ D_y(\zeta) &= (\lambda - u_x) D_x(\zeta), \end{cases} \quad (16)$$

while ζ_2 satisfies

$$\begin{cases} D_t(\zeta_2) &= (\lambda^2 - \lambda u_x - u_y) D_x(\zeta_2) + (\lambda u_{tx} + u_{ty}) \zeta_1 + (\lambda u_{xx} + u_{xy}) \zeta_2 \\ &\quad + (u_{tx} + u_y u_{xx} + (\lambda - u_x) u_{xy}) \zeta_3 - \lambda D_x(\varphi) - D_y(\varphi), \\ D_y(\zeta_2) &= (\lambda - u_x) D_x(\zeta_2) + u_{yx} \zeta_1 + u_{xx} \zeta_2 + u_{xy} \zeta_3 - D_x(\varphi), \end{cases} \quad (17)$$

The last system is compatible whenever φ is a symmetry of Eq. (1). Then we have

THEOREM 1. *The function $a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3$ is a nonlocal symmetry of Eq. (1) corresponding to the covering (3) whenever it is a \mathbb{R} -linear combination of the functions*

$$\psi = (\lambda^2 - \lambda u_x - u_y) \zeta_1 + \zeta_2 + (\lambda - u_x) \zeta_3$$

and

$$\eta = \frac{1}{q_x} \zeta, \quad (18)$$

where q satisfies (3) and ζ is a solution to (16).

REMARK 3. The coefficient at ζ in the r.h.s. of (18) satisfies the identity

$$\tilde{D}_y^2(v) = \tilde{D}_t \tilde{D}_x(v) + u_y \tilde{D}_x^2(v) + u_{xx} \tilde{D}_y(v) - u_x \tilde{D}_x \tilde{D}_y(v) - u_{xy} \tilde{D}_x(v).$$

Therefore, the function $1/q_x$ is a shadow of a nonlocal symmetry of Eq. (1) corresponding to the covering (3).

Since ζ_1, ζ_3 satisfy (16) and ζ_2 satisfies (17), then ψ is a solution to the following over-determined system

$$\begin{cases} D_t(\psi) &= (\lambda^2 - \lambda u_x - u_y) D_x(\psi) + (\lambda u_{xx} + u_{xy}) \zeta_2 \psi - \lambda D_x(\varphi) - D_y(\varphi), \\ D_y(\psi) &= (\lambda - u_x) D_x(\psi) + u_{xx} \psi - D_x(\varphi), \end{cases} \quad (19)$$

This system is compatible whenever φ is a symmetry of Eq. (1). Each solution ψ to this system is a symmetry of (1), too. Therefore, Eqs. (19) define a recursion operator $\psi = \mathcal{R}(\varphi)$ for (1). We express $D_x(\varphi)$ and $D_y(\varphi)$ from (19):

$$\begin{cases} D_x(\varphi) &= (\lambda - u_x) D_x(\psi) - D_y(\psi) + u_{xx} \psi, \\ D_y(\varphi) &= \lambda D_y(\psi) - D_t(\psi) - u_y D_x(\psi) + u_{xy} \psi. \end{cases} \quad (20)$$

This system is compatible whenever ψ is a symmetry of (1). Whence (20) defines the inverse recursion operator $\varphi = \mathcal{R}^{-1}(\psi)$. Both systems (19) and (20) define Bäcklund transformations for the tangent covering of Eq. (1).

4.2. rdDym equation

For Eq. (2) the computations are very similar to those of the previous subsection. Eqs. (14) yield two systems

$$\begin{cases} D_t(\zeta) &= (u_x - \lambda) D_x(\zeta), \\ D_y(\zeta) &= \lambda^{-1} u_y D_x(\zeta), \end{cases} \quad (21)$$

and

$$\begin{cases} D_t(\zeta_2) &= (u_x - \lambda) D_x(\zeta_2) - u_{tx} \zeta_1 - u_{xx} \zeta_2 - u_{xy} \zeta_3 + D_x(\varphi), \\ D_y(\zeta_2) &= \lambda^{-1} (u_y D_x(\zeta_2) - (u_x u_{xy} - u_y u_{xx}) \zeta_1 - u_{xy} \zeta_2 - u_{yy} \zeta_3 + D_y(\varphi)), \end{cases} \quad (22)$$

such that ζ_1 and ζ_3 are solutions to (22), while the system for ζ_2 is compatible whenever φ is a symmetry of (2). Then routine computations give

THEOREM 2. *The function $a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3$ is a nonlocal symmetry of Eq. (2) corresponding to the covering (4) if, and only if, it is a \mathbb{R} -linear combination of the functions*

$$\psi = (u_x - \lambda) \zeta_1 + \zeta_2 + \lambda^{-1} u_y \zeta_3 \quad (23)$$

and

$$\eta = \frac{1}{q_x} \zeta, \quad (24)$$

where q and ζ meet (4) and (21), respectively.

REMARK 4. The coefficient at ζ in the r.h.s. of (24) is a solution of the following equation:

$$\tilde{D}_t \tilde{D}_y(v) = u_x \tilde{D}_x \tilde{D}_y(v) + u_{xy} \tilde{D}_x(v) - u_y \tilde{D}_x^2(v) - u_{xx} \tilde{D}_y(v).$$

Therefore, the function $1/q_x$ is a shadow of a nonlocal symmetry of Eq. (2) corresponding to the covering (4).

The function (23) is a solution to the system

$$\begin{cases} D_t(\psi) &= (u_x - \lambda) D_x(\psi) - u_{xx} \psi + D_x(\varphi), \\ D_y(\psi) &= \lambda^{-1} (u_y D_x(\psi) - u_{xy} \psi + D_y(\varphi)), \end{cases} \quad (25)$$

This auto-Bäcklund transformation for the tangent covering of (2) defines the recursion operator $\psi = \mathcal{R}(\varphi)$. The inverse recursion operator $\varphi = \mathcal{R}^{-1}(\psi)$ is defined by the system

$$\begin{cases} D_t(\varphi) &= D_t(\psi) + (\lambda - u_x) D_x(\psi) + u_{xx} \psi \\ D_y(\varphi) &= \lambda D_y(\psi) - u_y D_x(\psi) + u_{xy} \psi. \end{cases} \quad (26)$$

5. Actions of recursion operators on contact symmetries

We consider actions of the recursion operators (19), (25) and their inverses (20), (26) to the contact symmetries of the corresponding Eqns. (1) and (2). The standard computational procedures [36, 49, 13, 18, 3, 33, 4] provide generators $\varphi \in \text{sym}_0(\mathcal{E})$. When φ are known, we solve Eqns. (19) and (25) for $\psi = \mathcal{R}(\varphi)$. To find actions of \mathcal{R}^{-1} on contact symmetries, we consider ψ in (20), (26) to be known elements of $\text{sym}_0(\mathcal{E})$ and then solve these systems for φ . For both Eqns. (1) and (2) it appears to be easier to find actions of \mathcal{R}^{-1} than ones of \mathcal{R} , so we start from \mathcal{R}^{-1} in both cases.

5.1. rmdkP equation

The infinitesimal generators of the contact symmetry algebra for Eq. (1) are

$$\varphi_0(A_0) = A_0 u_t + \frac{1}{2} (2 A'_0 x + A''_0 y^2) u_x + A'_0 y u_y - A'_0 u - A''_0 x y - \frac{1}{6} A'''_0 y^3,$$

$$\varphi_1(A_1) = A'_1 y u_x + A_1 u_y - A'_1 x - \frac{1}{2} A''_1 y^2,$$

$$\varphi_2(A_2) = A_2 u_x - A'_2 y,$$

$$\varphi_3(A_3) = A_3,$$

$$\varphi_4 = 2x u_x + y u_y - 3u, \quad \varphi_5 = y u_x - 2x,$$

where A_j together with B_k below are arbitrary smooth functions of the variable t . The commutators for these generators read

$$\begin{aligned} \{\varphi_j(A_j), \varphi_k(B_k)\} &= \begin{cases} \varphi_{j+k}(A_j B'_k - B_k A'_j), & 0 \leq j+k \leq 3, \\ 0, & j+k > 3, \end{cases} \\ \{\varphi_k(A_k), \varphi_4\} &= k \varphi_k(A_k), \quad 0 \leq k \leq 3, \\ \{\varphi_0(A_0), \varphi_5\} &= \{\varphi_3(A_3), \varphi_5\} = 0, \\ \{\varphi_1(A_1), \varphi_5\} &= \varphi_2(A_1), \\ \{\varphi_2(A_2), \varphi_5\} &= 2 \varphi_3(A_2), \\ \{\varphi_4, \varphi_5\} &= -\varphi_5. \end{aligned}$$

We denote by \mathfrak{g}_j the spans of $\varphi_j(A_j)$ when $A_j \in C^\infty(\mathbb{R})$, $0 \leq j \leq 3$, and put $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \mathbb{R} \varphi_4$, $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1 \oplus \mathbb{R} \varphi_5$, $\tilde{\mathfrak{g}}_2 = \mathfrak{g}_2$, $\tilde{\mathfrak{g}}_3 = \mathfrak{g}_3$. Then the contact symmetry algebra of (1) $\text{sym}_0(\mathcal{E}) = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2 \oplus \tilde{\mathfrak{g}}_3$ has the following grading

$$\{\tilde{\mathfrak{g}}_j, \tilde{\mathfrak{g}}_k\} = \begin{cases} \mathfrak{g}_0, & j+k=0, \\ \tilde{\mathfrak{g}}_{j+k}, & 0 < j+k \leq 3, \\ 0, & j+k > 3. \end{cases}$$

The solutions to Eqns. (20) are defined up to adding their arbitrary solution with $\psi = 0$, i.e., an arbitrary element of the subalgebra \mathfrak{g}_3 . We will not write these elements explicitly, or, in other words, we will consider factor spaces w.r.t. \mathfrak{g}_3 . For $\psi = \varphi_j(A_j)$ with $1 \leq j \leq 3$ and $\psi = \varphi_5$ we get local solutions

$$\begin{aligned} \mathcal{R}^{-1}(\varphi_1(A_1)) &= -\varphi_0(A_1) + \lambda \varphi_1(A_1), \\ \mathcal{R}^{-1}(\varphi_2(A_2)) &= -\varphi_1(A_2) + \lambda \varphi_2(A_2), \\ \mathcal{R}^{-1}(\varphi_3(A_3)) &= \varphi_2(A_3), \\ \mathcal{R}^{-1}(\varphi_5) &= -\varphi_4 + \lambda \varphi_5. \end{aligned}$$

Further, we have

$$\begin{aligned} \mathcal{R}^{-1}(\varphi_0(A_0)) &= s_0 + \frac{1}{24} y^4 A_0^{(iv)} + \frac{1}{6} y^2 (3x - y(u_x + \lambda)) A_0''' \\ &\quad + \frac{1}{2} (y(\lambda y - 2x) u_x - y^2 u_y + x^2 - 2\lambda x y) A_0'' \\ &\quad - (y u_t + (u - \lambda x) u_x + (x - \lambda y) u_y + 2u) A_0' + \lambda u_t A_0, \end{aligned} \quad (27)$$

where s_0 is a solution to the following compatible system

$$\begin{cases} s_{0,x} &= y u_x A_0'' + (u_x^2 + u_y) A_0' - (u_{ty} + u_x u_{tx} - u_t u_{xx}) A_0, \\ s_{0,y} &= (y u_y + u) A_0'' + (u_t + u_x u_y) A_0' - (u_{ty} + u_t u_{xy} - u_y u_{xx}) A_0. \end{cases} \quad (28)$$

In other words, s_0 is a potential of the conservation law

$$ds_0 \wedge dt = ((y u_y + u) A_0'' + (u_t + u_x u_y) A_0' - (u_{ty} + u_t u_{xy} - u_y u_{xx}) A_0) dx \wedge dt \\ + (y u_x A_0'' + (u_x^2 + u_y) A_0' - (u_{ty} + u_x u_{tx} - u_t u_{xx}) A_0) dy \wedge dt$$

of Eq. (1). For φ_4 we get

$$\mathcal{R}^{-1}(\varphi_4) = 4 s_4 - y u_t + (2 \lambda x - 3 u) u_x + (\lambda y - 2 x) u_y - 3 \lambda u, \quad (29)$$

where s_4 is a solution to

$$\begin{cases} s_{4,x} &= u_y + u_x^2, \\ s_{4,y} &= u_t + u_x u_y, \end{cases} \quad (30)$$

that is, a potential of the conservation law

$$ds_4 \wedge dt = (u_t + u_x u_y) dx \wedge dt + (u_y + u_x^2) dy \wedge dt.$$

Thus the action of \mathcal{R}^{-1} to $\varphi_0(A_0)$ and φ_4 provides shadows of nonlocal symmetries to Eq. (1): the infinite set (27) corresponds to the covering (28), and (29) corresponds to the covering (30).

Factorizing w.r.t. solutions of Eqs. (19) with $\varphi = 0$, we have

$$\mathcal{R}(\varphi_1(A_1)) = -\varphi_2(A_1) + \lambda \varphi_3(A_1),$$

$$\mathcal{R}(\varphi_2(A_2)) = \varphi_3(A_2),$$

$$\mathcal{R}(\varphi_3(A_3)) = 0,$$

while the solutions $\mathcal{R}(\varphi_0(A_0))$, $\mathcal{R}(\varphi_4)$, $\mathcal{R}(\varphi_5)$ of (19) with $\varphi = \varphi_0(A_0)$, $\varphi = \varphi_4$, and $\varphi = \varphi_5$, respectively, are shadows of new nonlocal symmetries of Eq. (1).

5.2. rdDym equation

The Lie algebra of contact symmetries of Eq. (2) is generated by

$$\varphi_0(A_0) = u_t A_0 + x u_x A_0' - u A_0' + \frac{1}{2} x^2 A_0'',$$

$$\varphi_1(A_1) = u_x A_1 + x A_1',$$

$$\varphi_2(A_2) = A_2,$$

$$\varphi_3(B_1) = u_y B_1,$$

$$\varphi_4 = x u_x - 2 u,$$

where A_j and C_k below are arbitrary functions of t , and B_j are arbitrary functions of y . The generators have the following commutators:

$$\{\varphi_j(A_j), \varphi_k(C_k)\} = \begin{cases} \varphi_{j+k}(A_j C_k' - C_k A_j'), & 0 \leq j+k \leq 2, \\ 0, & j+k > 2, \end{cases}$$

$$\{\varphi_j(A_j), \varphi_3(B_1)\} = 0, \quad 0 \leq j \leq 2,$$

$$\begin{aligned}\{\varphi_k(A_k), \varphi_4\} &= k \varphi_k(A_k), \quad 0 \leq k \leq 2, \\ \{\varphi_3(B_1), \varphi_3(B_2)\} &= \varphi_3(B_1 B'_2 - B_2 B'_1), \\ \{\varphi_3(B_1), \varphi_4\} &= 0\end{aligned}$$

With $\mathfrak{g}_j = \{\varphi_j(A_j) \mid A_j \in C^\infty(\mathbb{R})\}$, $0 \leq j \leq 2$, $\mathfrak{h} = \{\varphi_3(B) \mid B \in C^\infty(\mathbb{R})\}$, $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \mathbb{R} \varphi_4$, $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1$, $\tilde{\mathfrak{g}}_2 = \mathfrak{g}_2$, and $\mathfrak{g} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$ we have $\text{sym}_0(\mathcal{E}) = \mathfrak{g} \oplus \mathfrak{h}$, $\{\mathfrak{g}, \mathfrak{h}\} = 0$, $\{\mathfrak{h}, \mathfrak{h}\} = \mathfrak{h}$, while the subalgebra \mathfrak{g} has the following grading

$$\{\tilde{\mathfrak{g}}_j, \tilde{\mathfrak{g}}_k\} = \begin{cases} \mathfrak{g}_0, & j+k=0 \\ \tilde{\mathfrak{g}}_{j+k}, & 0 < j+k \leq 2, \\ 0, & j+k > 2. \end{cases}$$

Then up to adding arbitrary solutions of (25) we have

$$\begin{aligned}\mathcal{R}^{-1}(\varphi_1(A_1)) &= \varphi_0(A_1) + \lambda \varphi_1(A_1), \\ \mathcal{R}^{-1}(\varphi_2(A_2)) &= \varphi_1(A_2), \\ \mathcal{R}^{-1}(\varphi_3(B_1)) &= \lambda \varphi_3(B_1),\end{aligned}$$

while

$$\begin{aligned}\mathcal{R}^{-1}(\varphi_0(A_0)) &= s_0 + \frac{1}{6} x^3 A_0''' + \frac{1}{2} (x u_x - 2u + \lambda x) A_0'' \\ &\quad + (x(u_t + \lambda u_x) - u(u_x + \lambda)) A_0' + u_t(u_x + \lambda) A_0 + A_3,\end{aligned}\tag{31}$$

where s_0 is a solution of the system

$$\begin{cases} s_{0,x} &= A_0(u_{tt} - 2u_x u_{tx}) - A_0'(u_t - u_x^2), \\ s_{0,y} &= A_0' u_x u_y - A_1(u_y u_{tx} + u_x^2 u_{xy} - u_x u_y u_{xx}), \end{cases}\tag{32}$$

that is a potential of the conservation law

$$\begin{aligned}ds_0 \wedge dt &= (A_0' u_x u_y - A_1(u_y u_{tx} + u_x^2 u_{xy} - u_x u_y u_{xx})) dy \wedge dt \\ &\quad + (A_0(u_{tt} - 2u_x u_{tx}) - A_0'(u_t - u_x^2)) dx \wedge dt,\end{aligned}$$

and

$$\mathcal{R}^{-1}(\varphi_4) = 3s_4 + x u_t - 2u u_x - 2\lambda u\tag{33}$$

where s_4 meets

$$\begin{cases} s_{4,x} &= u_x^2 - u_t, \\ s_{4,y} &= u_x u_y, \end{cases}\tag{34}$$

and therefore defines a conservation law

$$ds_4 \wedge dt = (u_x^2 - u_t) dx \wedge dt + u_x u_y dy \wedge dt$$

of (2). Whence Eq. (2) has the infinite set of shadows of nonlocal symmetries (31) corresponding to the covering (32) and (33) corresponding to the covering (34). Also, factorizing w.r.t. arbitrary solutions of (26) we have

$$\mathcal{R}(\varphi_0(A_0)) = \varphi_1(A_0) - \lambda \varphi_2(A_0),$$

$$\mathcal{R}(\varphi_1(A_1)) = \varphi_2(A_1),$$

$$\mathcal{R}(\varphi_2(A_2)) = 0,$$

$$\mathcal{R}(\varphi_3(B_1)) = \lambda^{-1} \varphi_3(B_1),$$

while the solution $\mathcal{R}(\varphi_4)$ to (26) with $\varphi = \varphi_4$ is a shadow of new nonlocal symmetry of Eq. (2).

6. Conclusion

In this paper we used the construction of [30] to find recursion operators for integrable cases of the rmdKP and rdDym equations. As a byproduct of computations, we found shadows of nonlocal symmetries of these equations corresponding to their coverings with nonremovable parameters (Remarks 3 and 4). Also, we studied actions of the recursion operators to contact symmetry algebras of (1), (2) and found shadows of nonlocal symmetries corresponding to coverings generated by conservation laws. As it is noted in Remark 1, every shadow φ provides a nonlocal symmetry in the corresponding covering τ_φ of (1) and (2). The structure of the spaces of nonlocal symmetries of Eqns. (1) and (2) is a subject of a further study. Another promising field of research is an application of the useful method of [30] to other PDEs with linear coverings which have non-removable parameters. Also, the problem of finding recursion operators for nonlinear coverings of PDEs in more than two independent variables seems to be very interesting and important.

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